MATH 141 HOMEWORK 8 DUE DATE: MONDAY, JUNE 12TH

Name: Solutions

This homework is due Monday, June 12th during recitation. If you have questions regarding any of this, feel free to ask during office hours or send me an email. When writing solutions, present your answers clearly and neatly, showing only necessary work.

1. Find a formula for the Riemann sum obtained by dividing the interval $[a, b]$ into n equal subintervals and using the right-hand endpoint to determine the height of the rectangles. Then take a limit of these sums as $n \to \infty$ to calculate the area under the curve over $[a, b]$.

   (a) $f(x) = 3x + 1$ over the interval $[1, 2]$ $\Delta x_j = \frac{b-a}{n} = \frac{2-1}{n} = \frac{1}{n}$

$$\sum_{j=1}^{n} \Delta x_j f(a + j\Delta x_j) = \sum_{j=1}^{n} \frac{1}{n}\left(3(1 + j\tfrac{1}{n}) + 1\right) = \frac{1}{n}\sum_{j=1}^{n} 3 + \frac{3}{n}j + 1$$

$$= \frac{1}{n}\sum_{j=1}^{n} 4 + \frac{3}{n}j = \frac{1}{n}\sum_{j=1}^{n} 4 + \frac{1}{n}\sum_{j=1}^{n}\frac{3}{n}j = \frac{4}{n}\sum_{j=1}^{n} 1 + \frac{3}{n^2}\sum_{j=1}^{n} j$$

$$= \frac{4}{n}n + \frac{3}{n^2}\cdot\frac{n(n+1)}{2} = 4 + \frac{3(n+1)}{2n} = 4 + \frac{3n}{2n} + \frac{3}{2n}$$

$$= 4 + \frac{3}{2} + \frac{3}{2n} = \boxed{\frac{11}{2} + \frac{3}{2n}}$$

$$\lim_{n\to\infty} \frac{11}{2} + \frac{3}{2n} = \frac{11}{2} + \frac{3}{2}\lim_{n\to\infty}\frac{1}{n} = \frac{11}{2} + \frac{3}{2}(0)$$

$$= \boxed{\frac{11}{2}}$$

(b) $f(x) = 2x^2 + x + 2$ over the interval $[2, 4]$ $\Delta x_j = \frac{b-a}{n} = \frac{4-2}{n} = \frac{2}{n}$

$$\sum_{j=1}^{n} \Delta x_j f(a + j\Delta x_j) = \sum_{j=1}^{n} \frac{2}{n}\left(2\left(2+\frac{2}{n}j\right)^2 + \left(2+\frac{2}{n}j\right) + 2\right)$$

$$= \frac{2}{n}\sum_{j=1}^{n} 2\left(4 + \frac{8}{n}j + \frac{4}{n^2}j^2\right) + 4 + \frac{2}{n}j = \frac{2}{n}\sum_{j=1}^{n} 8 + \frac{16}{n}j + \frac{8}{n^2}j + 4 + \frac{2}{n}j$$

$$= \frac{2}{n}\sum_{j=1}^{n} 12 + \frac{18}{n}j + \frac{8}{n^2}j^2 = \frac{2}{n}\sum_{j=1}^{n} 12 + \frac{2}{n}\sum_{j=1}^{n}\frac{18}{n}j + \frac{2}{n}\sum_{j=1}^{n}\frac{8}{n^2}j^2$$

$$= \frac{24}{n}\sum_{j=1}^{n} 1 + \frac{36}{n^2}\sum_{j=1}^{n} j + \frac{16}{n^3}\sum_{j=1}^{n} j^2 = \frac{24}{n}\cdot n + \frac{36}{n^2}\cdot\frac{n(n+1)}{2} + \frac{16}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}$$

$$= 24 + \frac{18(n+1)}{n} + \frac{8(2n^2+3n+1)}{3n^2} = 24 + \frac{18n}{n} + \frac{18}{n} + \frac{8}{3}\left(\frac{2n^2}{n^2} + \frac{3n}{n^2} + \frac{1}{n^2}\right)$$

$$= 24 + 18 + \frac{18}{n} + \frac{16}{3} + \frac{8}{n} + \frac{8}{3n^2} = 42 + \frac{16}{3} + \frac{26}{n} + \frac{8}{3n^2}$$

$$= \boxed{\frac{142}{3} + \frac{26}{n} + \frac{8}{3n^2}}$$

$$\lim_{n\to\infty} \frac{142}{3} + \frac{26}{n} + \frac{8}{3n^2} = \frac{142}{3} + 26\lim_{n\to\infty}\frac{1}{n} + \frac{8}{3}\lim_{n\to\infty}\frac{1}{n^2}$$

$$= \frac{142}{3} + 26(0) + \frac{8}{3}(0) = \boxed{\frac{142}{3}}$$

2. Compute the following indefinite integrals.

(a) $\int x^4 + 3x^3 + 6x^2 + 8x + 2\, dx$

Answer: $\frac{1}{5}x^5 + \frac{3}{4}x^4 + 2x^3 + 4x^2 + 2x + C$

(b) $\int \frac{2x}{\sqrt{x^2+1}}\, dx$ $u = x^2 + 1$
$du = 2x\, dx$

$$= \int \frac{1}{\sqrt{u}}\, du = \int u^{-1/2}\, du$$

$$= \frac{u^{-1/2+1}}{-1/2+1} + C = 2u^{1/2} + C$$

Answer: $2(x^2+1)^{1/2} + C$

Cont.

(c) $\int 3\cos^2(x)\sin(x)\,dx$

$u = \cos(x)$
$du = -\sin(x)$

$= \int -3u^2 du = \frac{-3u^{2+1}}{2+1} + c = -u^3 + c$

Answer: $-\cos^3(x) + c$

3. Compute the following definite integrals.

(a) $\int_1^2 x^7 + 4x^3 + x + 2\,dx$

$= \frac{1}{8}x^8 + x^4 + \frac{1}{2}x^2 + 2x \Big|_1^2 = \frac{1}{8}2^8 + 2^4 + \frac{1}{2}2^2 + 2(2) - \frac{1}{8} - 1 - \frac{1}{2} - 2$

$= 2^5 + 2^4 + 2 + 4 - \frac{5}{8} - 3 = 32 + 16 + 3 - \frac{5}{8} = 51 - \frac{5}{8} = \frac{403}{8}$

Answer: $\frac{403}{8}$

(b) $\int_0^{\sqrt{3}} \frac{4x}{\sqrt{x^2+1}}\,dx$

$u = \cancel{3}x^2 + 1$
$du = 2x$

$\int \frac{2}{\sqrt{u}} du = \int 2u^{-1/2} du = \frac{2u^{-1/2+1}}{-1/2+1} = \frac{2u^{1/2}}{1/2} = 4u^{1/2}$

$= 4(x^2+1)^{1/2} \Big|_0^{\sqrt{3}} = 4(3+1)^{1/2} - 4(0+1)^{1/2} = 4(2) - 4 = 4$

Answer: 4

(c) $\int_0^{\pi/4} \tan(x)\sec^2(x)\,dx$

$u = \tan(x)$
$du = \sec^2(x)dx$

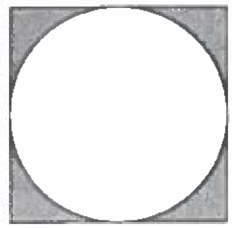
$\int u\,du = \frac{1}{2}u^2 = \frac{1}{2}\tan^2(x) \Big|_0^{\pi/4} = \frac{1}{2}\tan^2(\pi/4) - \frac{1}{2}\tan^2(0)$

$= \frac{1}{2}1^2 - \frac{1}{2}0^2 = \frac{1}{2}$

Answer: $1/2$.

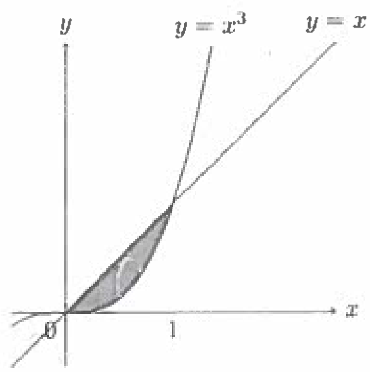
4. **Area Between Curves**

(a) Consider the diagram below. Explain, in words, how would you calculate the area of the shaded region given that the side length of the square is $2cm$?



Calculate the area of the square and subtract the area of the circle.

(b) Using the same idea as above, calculate the shaded region below.



$$\int_0^1 x\,dx - \int_0^1 x^3\,dx = \int_0^1 x - x^3\,dx$$

$$= \frac{1}{2}x^2 - \frac{1}{4}x^4 \Big|_0^1 = \frac{1}{2} - \frac{1}{4} - 0 = 1/4$$

Answer: 1/4

Cont.

5. (a) Compute the area between the curve $y = \tan(x)$ and the x-axis from $x = -\pi/4$ to $x = \pi/3$. Hint: Write $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$

$$\int_{-\pi/4}^{\pi/3} \tan(x)\,dx - \int_{-\pi/4}^{\pi/3} 0\,dx = \int_{-\pi/4}^{\pi/3} \frac{\sin(x)}{\cos(x)}\,dx \qquad u=\cos(x) \quad -du=\sin(x)dx$$

$$= -\int \frac{1}{u}\,du = -\ln|u| = -\ln|\cos(x)|\Big|_{-\pi/4}^{\pi/3}$$

$$= -\ln|\cos(\pi/3)| + \ln|\cos(-\pi/4)| = -\ln(1/2) + \ln(\sqrt{2}/2)$$

$$= \ln\left(\frac{\sqrt{2}/2}{1/2}\right) = \ln(\sqrt{2}) = \ln(2^{1/2}) = \frac{1}{2}\ln(2)$$

Answer: $\dfrac{\ln(2)}{2}$.

(b) Compute the area between the curves $y = 4 - 4x^2$ and $y = x^4 - 1$. Hint: To find the range of x values to integrate over, find the two points at which the two curves intersect.

$$4 - 4x^2 = x^4 - 1 \implies 4(1-x^2) = (x^4-1)$$

$$\implies 4(1-x^2) = -(1-x^2)(x^2+1)$$

$x \neq \pm 1 \implies 4 = -(x^2+1) \implies$ no solutions

$\implies x = \pm 1$ are the only points of intersection.

$$\int_{-1}^{1} 4 - 4x^2\,dx - \int_{-1}^{1} x^4 - 1\,dx = \int_{-1}^{1} -x^4 - 4x^2 + 5\,dx$$

$$= -\frac{1}{5}x^5 - \frac{4}{3}x^3 + 5x\Big|_{-1}^{1} = -\frac{1}{5} - \frac{4}{3} + 5 - \left(\frac{1}{5} + \frac{4}{3} - 5\right)$$

$$= -\frac{2}{5} - \frac{8}{3} + 10 = \frac{-6-40+150}{15}$$

Answer: $\dfrac{104}{15}$

The End.